FORMAL GAGA ON ARTIN STACKS

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Our aim is to explain how to use Olsson's theorem on proper hypercoverings of Artin stacks (the main result in [4]) to prove the formal GAGA theorem and the Grothendieck existence theorem for proper Artin stacks over an adic noetherian ring. In [4] a proof of this result is given from a different point of view. Everything in the first two sections below is definitions and "general nonsense".

1. Completion of coherent sheaves and stacks

Let $\mathscr{X}$ be a locally noetherian stack and $X_0 \subseteq |\mathscr{X}|$ a closed subset.

Definition 1.1. For $\mathscr{F} \in \mathrm{Coh}(\mathscr{X})$, the *completion of* $\mathscr{F}$ *along* X_0 is the sheaf

$$\widehat{\mathscr{F}} = \varprojlim_{\alpha} \mathscr{F} / \mathscr{J}_\alpha \mathscr{F}$$

on $\mathscr{X}_{\text{lis-ét}}$, where $\{\mathscr{J}_\alpha\}$ is the inverse system of coherent ideals on $\mathscr{X}$ with zero locus X_0. Define the sheaf of rings

$$\mathscr{O}_{\widehat{\mathscr{X}}} \stackrel{\text{def}}{=} \widehat{\mathscr{O}_{\mathscr{X}}} = \varprojlim_{\alpha} \mathscr{O}_{\mathscr{X}} / \mathscr{J}_\alpha.$$

Note that $\mathscr{O}_{\widehat{\mathscr{X}}}$ is a flat sheaf of algebras on $\mathscr{X}_{\text{lis-ét}}$ (in the sense of [3, 12.7]), and $\widehat{\mathscr{F}}$ is naturally an $\mathscr{O}_{\widehat{\mathscr{X}}}$-module for $\mathscr{F} \in \mathrm{Coh}(\mathscr{X})$.

Definition 1.2. The ringed topos $\widehat{\mathscr{X}}$ is the site $\mathscr{X}_{\text{lis-ét}}$ equipped with the sheaf of rings $\mathscr{O}_{\widehat{\mathscr{X}}}$. This is called the *completion of* $\mathscr{X}$ *along* X_0.

Suppose $\mathscr{X}$ is a locally noetherian Deligne–Mumford stack. Definition 1.2 has an obvious variant $\widehat{\mathscr{X}}_{\text{ét}}$ using the underlying smaller étale site $\mathscr{X}_{\text{ét}}$ and the restriction $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}$ of $\mathscr{O}_{\widehat{\mathscr{X}}}$ to this site. By [3, 12.7.4], the category of cartesian $\mathscr{O}_{\widehat{\mathscr{X}}}$-modules on $\mathscr{X}_{\text{lis-ét}}$ is equivalent to the category of $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}$-modules on $\mathscr{X}_{\text{ét}}$:

$$\mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}},\text{cart}}(\mathscr{O}_{\widehat{\mathscr{X}}}) \simeq \mathrm{Mod}_{\mathscr{X}_{\text{ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}) \tag{1.1}$$

Definition 1.3. Let $\mathscr{X}$ be a locally noetherian stack and $X_0 \subseteq |\mathscr{X}|$ a closed subset, and let $\widehat{\mathscr{X}}$ be the completion of $\mathscr{X}$ along X_0. The category $\mathrm{Coh}(\widehat{\mathscr{X}})$ of *coherent sheaves* is the full subcategory of cartesian $\mathscr{O}_{\widehat{\mathscr{X}}}$-modules on $\mathscr{X}_{\text{lis-ét}}$ that are locally of finite presentation.

If $\mathscr{X}$ is a locally noetherian Deligne–Mumford stack then the category $\mathrm{Coh}(\widehat{\mathscr{X}}_{\text{ét}})$ of *coherent sheaves* is the full subcategory of $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}$-modules on $\mathscr{X}_{\text{ét}}$ that are locally of finite presentation.

Remark 1.4. When $\mathscr{X}$ is a locally noetherian Deligne–Mumford stack, the equivalence (1.1) restricts to an equivalence of categories between $\mathrm{Coh}(\widehat{\mathscr{X}})$ and $\mathrm{Coh}(\widehat{\mathscr{X}}_{\text{ét}})$ since $\mathscr{O}_{\widehat{\mathscr{X}}}$ is a flat sheaf of rings on $\mathscr{X}_{\text{lis-ét}}$.

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The functoriality of Definition 1.2 is straightforward, as follows. Let $f : \mathscr{X}' \to \mathscr{X}$ be a 1-morphism between locally noetherian stacks, where we specify closed subsets $X_0 \subseteq |\mathscr{X}|$ and $X_0' \subseteq |\mathscr{X}'|$ with $X_0' \subseteq |f|^{-1}(X_0)$. The map of ringed topoi

$$(f_*, f^*) : \mathscr{X}' \to \mathscr{X}$$

and the natural map $f^*(\mathscr{O}_{\widehat{\mathscr{X}}}) \to \mathscr{O}_{\widehat{\mathscr{X}'}}$ induce a map of ringed topoi

$$\widehat{f} : \widehat{\mathscr{X}'} \to \widehat{\mathscr{X}}$$

such that $\widehat{f}^*$ preserves coherence. Similarly, when $\mathscr{X}$ and $\mathscr{X}'$ are locally noetherian Deligne–Mumford stacks we have a natural map of ringed topoi

$$\widehat{f}_{\text{ét}} : \widehat{\mathscr{X}'}_{\text{ét}} \to \widehat{\mathscr{X}}_{\text{ét}}$$

that is compatible with $\widehat{f}$, and $\widehat{f}^*_{\text{ét}}$ preserves coherence.

Since formal locally noetherian algebraic spaces have an étale topology, we must compare Definition 1.2 with formal completion of locally noetherian algebraic spaces. To do this, let $\mathscr{X}$ be a locally noetherian algebraic space, and $X_0 \subseteq |\mathscr{X}|$ a closed subset. The formal completion $\widehat{\mathscr{X}}_{\text{formal}}$ along X_0 is the category of sheaves of sets on the site $\mathscr{X}_{0,\text{ét}}$, and it is equipped with the sheaf of rings

$$\mathscr{O}_{\widehat{\mathscr{X}}_{\text{formal}}} : U_0 \rightsquigarrow \varprojlim \Gamma(U_n, \mathscr{O}_{U_n})$$

where $U_n \to \mathscr{X}_n$ is the unique étale algebraic space lifting $U_0 \to \mathscr{X}_0$. There is a natural map of topoi (ignoring structure sheaves!)

$$I_{\mathscr{X}} : \widehat{\mathscr{X}}_{\text{formal}} \to \mathscr{X}_{\text{ét}}.$$

Theorem 1.5. *Let $\mathscr{X}$ be a locally noetherian algebraic space, $X_0 \subseteq |\mathscr{X}|$ a closed subset. Considering $\mathscr{X}$ as a stack, let $\widehat{\mathscr{X}}$ the completion of $\mathscr{X}$ along X_0. The pullback functor*

$$I_{\mathscr{X}}^* : \mathrm{Ab}(\mathscr{X}_{\text{ét}}) \to \mathrm{Ab}(\widehat{\mathscr{X}}_{\text{formal}})$$

on categories of abelian sheaves restricts to an equivalence of categories

$$I_{\mathscr{X}}^* : \mathrm{Mod}_{\mathscr{X}_{\text{ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}) \to \mathrm{Mod}_{\mathscr{X}_{0,\text{ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{formal}}}) \tag{1.2}$$

Under this equivalence, the subcategory $\mathrm{Coh}(\widehat{\mathscr{X}}_{\text{ét}}) \simeq \mathrm{Coh}(\widehat{\mathscr{X}})$ goes over to the subcategory $\mathrm{Coh}(\widehat{\mathscr{X}}_{\text{formal}})$.

We emphasize that the pullback functor $I_{\mathscr{X}}^*$ in the theorem is on the level of abelian sheaves, not sheaves of modules over $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}$ or $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{formal}}}$.

Proof. Let

$$\mathscr{X}_0 \overset{i}{\hookrightarrow} \mathscr{X} \overset{j}{\hookleftarrow} \mathscr{U}$$

be the usual stratification. Clearly $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}|_{\mathscr{U}_{\text{ét}}} = 0$ as an abelian sheaf on $\mathscr{U}_{\text{ét}}$, so every $\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}$-module restricts to 0 as an abelian sheaf on $\mathscr{U}_{\text{ét}}$. Since $I_{\mathscr{X}}^*$ sets up an equivalence of categories between $\mathrm{Ab}(\widehat{\mathscr{X}}_{\text{formal}}) = \mathrm{Ab}(\mathscr{X}_{0,\text{ét}})$ and the category of abelian sheaves on $\mathscr{X}_{\text{ét}}$ that restrict to 0 over the open complement $\mathscr{U}$ of $\mathscr{X}_0$, we are reduced to proving that the natural map

$$I_{\mathscr{X}}^*(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}}) \to \mathscr{O}_{\widehat{\mathscr{X}}_{\text{formal}}} \tag{1.3}$$

of abelian sheaves on $\mathscr{X}_{0,\text{ét}}$, defined via adjunction, is an isomorphism.

We may work étale locally on $\mathscr{X}_{\text{ét}}$, and so we may reduce to the case when $\mathscr{X} = \operatorname{Spec} B$ is an affine scheme, say with ideal of definition I. It suffices to check that the map induced by (1.3) on stalks at geometric points $\overline{x}$ of the closed subscheme $\operatorname{Spec} B/I$ in $\operatorname{Spec} B$ are isomorphisms. This stalk map is the natural map of rings

$$\varinjlim_{U} \mathscr{O}(U)^{\wedge} \to \varinjlim_{V_0} \varprojlim_{n} \mathscr{O}(V_n) \tag{1.4}$$

where the left side is the direct limit of I-adic completions taken over the directed system of connected affine étale neighborhoods $U \to \operatorname{Spec} B$ of $\overline{x}$ (the rigidity of connected étale neighborhoods of a geometric point ensures that that U's form a directed system of pointed étale objects and not merely a diagram of such objects) and the right side is the direct limit over the directed system of connected affine étale neighborhoods $V_0 \to \mathscr{X}_0 = \operatorname{Spec} B/I$ of $\overline{x}$ with $V_n \to \mathscr{X}_n$ the unique étale lift of $V_0 \to \mathscr{X}_0$.

Since local schemes are connected and noetherian rings contain only finitely many idempotents, denominator-chasing shows that for each U, there exists a connected affine Zariski-open $\widetilde{U} \subseteq U$ containing $\overline{x}$ such that $\widetilde{U}_0$ is connected. Thus, the direct limit on the left side of (1.4) may be computed by working over the cofinal set of U's such that U_0 is also connected. Since $\mathscr{O}(U)^{\wedge} = \varprojlim \mathscr{O}(U_n)$ where the reduction $U_n \to \operatorname{Spec} B/I^{n+1}$ coincides with the unique étale lifting of $U_0 \to \operatorname{Spec} B/I$, the direct limit over U's with connected U_0 on the left side of (1.4) can be viewed as a restricted version of the direct limit over V_0's on the right side of (1.4). We therefore merely have to check cofinality of the U_0's in the V_0's, so we need to prove that a cofinal system of étale neighborhoods of $\overline{x}$ in $\operatorname{Spec} B/I$ is given by the reductions U_0 of connected affine étale neighborhoods U of $\overline{x}$ in $\operatorname{Spec} B$ such that the affine U_0 is *connected*.

By standard smearing-out arguments (as in the denominator-chasing used above), we may replace B with its algebraic localization under $\overline{x}$ and may work with local-étale neighborhoods of $\overline{x}$. This allows us to ignore the connectivity property, so we have to prove that every local-étale neighborhood of the geometric point $\overline{x}$ over the closed point in $\operatorname{Spec} B/I$ is refined by the reduction of a local-étale neighborhood of the geometric point $\overline{x}$ over the closed point in $\operatorname{Spec} B$. Passing to limits over local-étale neighborhoods, we may conclude by using that the unique B-algebra map $B_{\overline{x}}^{\text{sh}} \to (B/I)_{\overline{x}}^{\text{sh}}$ respecting $\overline{x}$ induces a map

$$B_{\overline{x}}^{\text{sh}}/IB_{\overline{x}}^{\text{sh}} \to (B/I)_{\overline{x}}^{\text{sh}} \tag{1.5}$$

that is an isomorphism [1, IV_4, 18.8.10]. ■

Remark 1.6. When $\mathscr{X}$ is a locally noetherian scheme, the formal scheme $\widehat{\mathscr{X}}_{\text{formal}}$ comes equipped with an étale ringed topos of its own, coinciding with the one associated to $\mathscr{X}$ considered as an algebraic space, so Theorem 1.5 and the natural equivalence between the categories of coherent sheaves on the étale and Zariski sites of $\widehat{\mathscr{X}}_{\text{formal}}$ give a natural equivalence $\operatorname{Coh}(\widehat{\mathscr{X}}) \simeq \operatorname{Coh}(\widehat{\mathscr{X}}_{\text{ét}}) \simeq \operatorname{Coh}(\widehat{\mathscr{X}}_{\text{formal,Zar}})$.

Corollary 1.7. *Let $\mathscr{X}$ be a locally noetherian stack, $X_0 \subseteq |\mathscr{X}|$ a closed subset, and $\widehat{\mathscr{X}}$ the completion of $\mathscr{X}$ along X_0. The category $\operatorname{Coh}(\widehat{\mathscr{X}})$ is abelian and as a full subcategory of $\operatorname{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}})$ it is stable under formation of kernels, cokernels, and extensions.*

Proof. Note that the full subcategory of cartesian objects in $\operatorname{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}})$ is stable under formation of kernels, cokernels, and extensions in $\operatorname{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}})$ since $\mathscr{O}_{\widehat{\mathscr{X}}}$ is a flat sheaf of rings. The definition of coherence is preserved under lisse-étale base change and for cartesian $\mathscr{O}_{\widehat{\mathscr{X}}}$-modules the property of coherence is local for the lisse-étale topology. We may therefore suppose that $\mathscr{X}$ is an affine scheme. By Remark 1.4 we are reduced to studying $\mathrm{Coh}(\widehat{\mathscr{X}}_{\text{ét}})$ as a subcategory of $\mathrm{Mod}_{\mathscr{X}_{\text{ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}})$. Theorem 1.5 translates this problem into that of studying $\mathrm{Coh}(\widehat{\mathscr{X}}_{\text{formal,ét}})$ as a subcategory of $\mathrm{Mod}(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{formal,ét}}})$, a situation for which all of the desired properties are known from the theory of formal schemes. ■

2. Equivalence of coherent sheaves and adic systems

Let $\mathscr{X}$ be a locally noetherian stack, $X_0 \subseteq |\mathscr{X}|$ a closed subset, and $\widehat{\mathscr{X}}$ the completion of $\mathscr{X}$ along X_0. By working locally, we see that for the canonical adjoint pair

$$\iota_{\mathscr{X}} = (\iota_{\mathscr{X}*}, \iota_{\mathscr{X}}^*) : \mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}}) \to \mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\mathscr{X}})$$

the functor $\iota_{\mathscr{X}}^*$ is exact (as this is obvious when $\mathscr{X}$ is a scheme with the lisse-étale topology, by consideration of strictly henselian stalks at geometric points). For any $\mathscr{F} \in \mathrm{Coh}(\mathscr{X})$ there is a canonical map $\mathscr{F} \to \iota_{\mathscr{X}*}(\widehat{\mathscr{F}})$ (with $\widehat{\mathscr{F}}$ as in Definition 1.1), and the adjoint

$$\iota_{\mathscr{X}}^*(\mathscr{F}) \to \widehat{\mathscr{F}} \tag{2.1}$$

is an isomorphism in $\mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}})$ because exactness of $\iota_{\mathscr{X}}^*$ and local considerations reduce us to the easy case $\mathscr{F} = \mathscr{O}_{\mathscr{X}}$. In particular, $\iota_{\mathscr{X}}^*$ preserves coherence.

Let $\mathscr{I}$ be an arbitrary coherent ideal on $\mathscr{X}$ with zero locus X_0 on $|\mathscr{X}|$, so $\mathscr{O}_{\widehat{\mathscr{X}}} \simeq \varprojlim \mathscr{O}_{\mathscr{X}}/\mathscr{I}^{n+1}$ on $\mathscr{X}_{\text{lis-ét}}$. It is clear that for $\mathscr{F} \in \mathrm{Coh}(\mathscr{X})$, there is a natural isomorphism of $\mathscr{O}_{\widehat{\mathscr{X}}}$-modules $\widehat{\mathscr{F}} \simeq \varprojlim \mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}$. This "$\mathscr{I}$-adic" description leads to:

Definition 2.1. Let $\mathscr{X}$ be a locally noetherian stack, and $\mathscr{I}$ a coherent ideal on $\mathscr{X}$. The category $\mathrm{Adic}_{\mathscr{I}}(\mathscr{X})$ is the category of projective systems $\mathscr{F}_\bullet = (\mathscr{F}_n)_{n \geq 0}$ of coherent $\mathscr{O}_{\mathscr{X}}$-modules such that $\mathscr{F}_n$ is killed by $\mathscr{I}^{n+1}$ and the map $\mathscr{F}_{n+1}/\mathscr{I}^{n+1}\mathscr{F}_{n+1} \to \mathscr{F}_n$ is an isomorphism for all $n \geq 0$. Morphisms are maps of projective systems. For $\mathscr{F}_\bullet \in \mathrm{Adic}_{\mathscr{I}}(\mathscr{X})$, the *completion* of $\mathscr{F}_\bullet$ is

$$\widehat{\mathscr{F}_\bullet} \stackrel{\text{def}}{=} \varprojlim \mathscr{F}_n \in \mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}}).$$

Lemma 2.2. *For $\mathscr{F}_\bullet \in \mathrm{Adic}_{\mathscr{I}}(\mathscr{X})$, the $\mathscr{O}_{\widehat{\mathscr{X}}}$-module $\widehat{\mathscr{F}_\bullet}$ is coherent.*

Proof. We first check that $\widehat{\mathscr{F}_\bullet}$ is cartesian in $\mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}})$. That is, for a smooth map $(f, \phi) : (U, u) \to (V, v)$ in $\mathscr{X}_{\text{lis-ét}}$, we must prove that the pullback map $\widehat{f}_{\text{ét}}^*(\widehat{\mathscr{F}}_{\bullet,(V,v)}) \to \widehat{\mathscr{F}}_{\bullet,(U,u)}$ of $\mathscr{O}_{\widehat{U}_{\text{ét}}}$-modules on $U_{\text{ét}}$ is an isomorphism. This is a trivial calculation on stalks at geometric points of $U_0 \subseteq U$, since the $\mathscr{F}_n$'s are coherent (and in particular, the $\mathscr{F}_n$'s are cartesian $\mathscr{O}_{\mathscr{X}}$-modules on $\mathscr{X}_{\text{lis-ét}}$).

Since the $\mathscr{O}_{\widehat{\mathscr{X}}}$-module $\widehat{\mathscr{F}_\bullet}$ on $\mathscr{X}_{\text{lis-ét}}$ is now proved to be cartesian, to verify its coherence we may use Remark 1.4 and Lemma 2.2 to work locally and with the étale topology. More specifically, it suffices to work in $\mathrm{Mod}(\mathscr{O}_{\widehat{\mathscr{X}}_{\text{ét}}})$ with $\mathscr{X} = \mathrm{Spec}\, A$ an affine noetherian scheme. In this case $\mathrm{Adic}_{\mathscr{I}}(\mathscr{X}) = \mathrm{Coh}(\mathrm{Spec}\, \widehat{A})$ using the Zariski topology on $\mathrm{Spec}\, \widehat{A}$, so via Theorem 1.5 and Remark 1.6 we just have to note that (2.1) is an isomorphism and that the pullback map $\mathrm{Coh}(\mathrm{Spec}\, \widehat{A}) \to \mathrm{Coh}(\mathrm{Spf}\, \widehat{A})$ using Zariski topologies is an equivalence of categories. ■

By Lemma 2.2, we see that completion defines a functor $\mathrm{Adic}_{\mathscr{I}}(\mathscr{X}) \to \mathrm{Coh}(\widehat{\mathscr{X}})$. Working locally for the lisse-étale topology, the natural map $\mathscr{O}_{\widehat{\mathscr{X}}}/\mathscr{I}^{n+1}\mathscr{O}_{\widehat{\mathscr{X}}} \to \mathscr{O}_{\mathscr{X}_n}$ is seen to be an isomorphism for all $n \geq 0$. Thus, $\mathscr{M}/\mathscr{I}^{n+1}\mathscr{M} \in \mathrm{Coh}(\mathscr{X}_n)$ for all $\mathscr{M} \in \mathrm{Coh}(\widehat{\mathscr{X}})$ and all $n \geq 0$, so we get a functor $\mathrm{Coh}(\widehat{\mathscr{X}}) \to \mathrm{Adic}_{\mathscr{I}}(\mathscr{X})$ via $\mathscr{M} \rightsquigarrow (\mathscr{M}/\mathscr{I}^{n+1}\mathscr{M})$.

Theorem 2.3. *Let $\mathscr{X}$ be a locally noetherian stack, $X_0 \subseteq |\mathscr{X}|$ a closed subset, and $\widehat{\mathscr{X}}$ the completion of $\mathscr{X}$ along X_0. Let $\mathscr{I}$ be a coherent ideal on $\mathscr{X}$ with zero locus X_0 on $|\mathscr{X}|$. The functors*

$$T_1 : \mathrm{Adic}_{\mathscr{I}}(\mathscr{X}) \to \mathrm{Coh}(\widehat{\mathscr{X}}),\ \ T_2 : \mathrm{Coh}(\widehat{\mathscr{X}}) \to \mathrm{Adic}_{\mathscr{I}}(\mathscr{X})$$

defined by $T_1(\mathscr{F}_\bullet) = \widehat{\mathscr{F}_\bullet} = \varprojlim \mathscr{F}_n$ and $T_2(\mathscr{M}) = (\mathscr{M}/\mathscr{I}^{n+1}\mathscr{M})$ are quasi-inverse equivalences of categories.

Proof. There are evident natural transformations $T_2 \circ T_1 \to \mathrm{id}$ and $\mathrm{id} \to T_1 \circ T_2$. To prove that these are isomorphisms we can work lisse-étale locally. Thus, as in the proof of Lemma 2.2, we may suppose $\mathscr{X} = \mathrm{Spec}\, A$ is affine and we may work in the étale topology. Obviously $\mathrm{Adic}_{\mathscr{I}}((\mathrm{Spec}\, A)_{\text{ét}}) = \mathrm{Coh}((\mathrm{Spec}\, \widehat{A})_{\mathrm{Zar}})$ with $\widehat{A}$ the completion of A with respect to the ideal $I = \Gamma(\mathscr{X}, \mathscr{I})$, and by Remark 1.6 we have $\mathrm{Coh}(\widehat{\mathscr{X}}) \simeq \mathrm{Coh}(\mathrm{Spf}\, \widehat{A})$. It is a basic fact in the theory of formal noetherian schemes that the map of ringed spaces $\mathrm{Spf}\, \widehat{A} \to \mathrm{Spec}\, \widehat{A}$ induces an equivalence of categories of coherent sheaves via ringed-space pullback, and this gives the desired result. ∎

3. Formal GAGA for stacks

Let $\mathscr{X}$ be a locally noetherian stack, $X_0 \subseteq |\mathscr{X}|$ a closed subset, and $\widehat{\mathscr{X}}$ the completion of $\mathscr{X}$ along X_0. Let $\iota_{\mathscr{X}} : \widehat{\mathscr{X}} \to \mathscr{X}$ be the canonical flat map of ringed topoi. Since the functor $\iota_{\mathscr{X}}^* : \mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\mathscr{X}}) \to \mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\widehat{\mathscr{X}}})$ is exact and preserves coherence, it defines a functor

$$\iota_{\mathscr{X}}^* : D^+(\mathscr{X}) \to D^+(\widehat{\mathscr{X}}) \tag{3.1}$$

that is natural in $\mathscr{X}$ and carries $D^+_{\mathrm{coh}}(\mathscr{X})$ into $D^+_{\mathrm{coh}}(\widehat{\mathscr{X}})$. We shall denote this functor on D^+'s as $\mathscr{F} \rightsquigarrow \widehat{\mathscr{F}}$, a notation that is consistent with our earlier notion of completion of coherent sheaves in Definition 1.1 because (2.1) is an isomorphism.

Now let $f : \mathscr{X} \to \mathscr{S}$ be a map of locally noetherian stacks. Choose closed subsets $S_0 \subseteq |\mathscr{S}|$ and $X_0 \subseteq |\mathscr{X}|$ such that $X_0 \subseteq |f|^{-1}(S_0)$. We write $\widehat{\mathscr{S}}$ and $\widehat{\mathscr{X}}$ to denote the respective completions of $\mathscr{S}$ and $\mathscr{X}$ along S_0 and X_0, and let $\widehat{f} : \widehat{\mathscr{X}} \to \widehat{\mathscr{S}}$ denote the canonical map induced by f. Since the diagram

$$\begin{array}{ccc} \widehat{\mathscr{X}} & \xrightarrow{\widehat{f}} & \widehat{\mathscr{S}} \\ {\scriptstyle \iota_{\mathscr{X}}}\downarrow & & \downarrow{\scriptstyle \iota_{\mathscr{S}}} \\ \mathscr{X} & \xrightarrow[f]{} & \mathscr{S} \end{array}$$

commutes, for $\mathscr{F}$ in $D^+(\mathscr{X})$ there is a canonical (*relative*) *comparison morphism*

$$(\mathbf{R}f_*(\mathscr{F}))^\wedge \to \mathbf{R}\widehat{f}_*(\widehat{\mathscr{F}})$$

in $D^+(\widehat{\mathscr{S}})$. In the special case that $\mathscr{S} = \mathrm{Spec}\, A$ is affine, with $\widehat{A}$ the completion of A corresponding to the choice of closed subset $S_0 \subseteq |\mathscr{S}| = \mathrm{Spec}\, A$, we may also define a canonical (*affine*) *comparison morphism*

$$\widehat{A} \otimes_A \mathbf{R}\Gamma(\mathscr{X}, \mathscr{F}) \to \mathbf{R}\Gamma(\widehat{\mathscr{X}}, \widehat{\mathscr{F}})$$

in $D^+(\mathrm{Mod}(\widehat{A}))$ for all $\mathscr{F}$ in $D^+(\mathscr{X})$.

Theorem 3.1 (relative formal GAGA)**.** *Let $f : \mathscr{X} \to \mathscr{S}$ be a proper* 1*-morphism of locally noetherian stacks. Let $S_0 \subseteq |\mathscr{S}|$ and $X_0 \subseteq |\mathscr{X}|$ be closed subsets with $|f|^{-1}(S_0) = X_0$, and let $\widehat{\mathscr{S}}$ and $\widehat{\mathscr{X}}$ be the corresponding completions of $\mathscr{S}$ and $\mathscr{X}$ along S_0 and X_0.*

The relative comparison morphism

$$(\mathbf{R}f_*(\mathscr{F}))^\wedge \to \mathbf{R}\widehat{f}_*(\widehat{\mathscr{F}}) \tag{3.2}$$

in $D^+(\widehat{\mathscr{S}})$ is an isomorphism for all $\mathscr{F}$ in $D^+_{\mathrm{coh}}(\mathscr{X})$.

Corollary 3.2 (affine formal GAGA)**.** *Let A be a noetherian ring with an adic topology and completion $\widehat{A}$, and let $\mathscr{X}$ be a stack equipped with a proper* 1*-morphism $f : \mathscr{X} \to \operatorname{Spec} A$. Let $\widehat{\mathscr{X}}$ be the completion of $\mathscr{X}$ along the zero locus in $|\mathscr{X}|$ for an ideal of definition of A. For any $\mathscr{F} \in D^+_{\mathrm{coh}}(\mathscr{X})$, the affine comparison morphism*

$$\widehat{A} \otimes_A \mathbf{R}\Gamma(\mathscr{X}, \mathscr{F}) \to \mathbf{R}\Gamma(\widehat{\mathscr{X}}, \widehat{\mathscr{F}}) \tag{3.3}$$

is an isomorphism in $D^+(\mathrm{Mod}(\widehat{A}))$.

By standard Leray and localization arguments (on the base), to prove Theorem 3.1 for a specific f it is sufficient to prove Corollary 3.2 for the restrictions of $\mathscr{X}$ over affines that form a smooth cover of $\mathscr{S}$. Conversely, Theorem 3.1 for a proper 1-morphism $f : \mathscr{X} \to \mathscr{S} = \operatorname{Spec} A$ to an affine target implies Corollary 3.2 for the same 1-morphism. In particular, the theorem holds in the case of proper 1-morphisms f that are representable (by algebraic spaces) because Theorem 1.5 identifies the corollary for a proper algebraic space $\mathscr{X}$ with Knutson's formal GAGA theorem.

To prove the theorem in the general case, it suffices to prove the corollary in general. Observe that for a closed substack $I : \mathscr{Y} \hookrightarrow \mathscr{X}$ and $\mathscr{G} \in D^+_{\mathrm{coh}}(\mathscr{Y})$, the corollary for $\mathscr{G}$ on the proper A-stack $\mathscr{Y}$ is equivalent to the corollary for $I_*\mathscr{G}$ on the proper A-stack $\mathscr{X}$. Thus, *devissage* for the category of coherent sheaves on a noetherian stack (see [3, 15.7]) reduces us to considering the case when $\mathscr{X}$ is integral and proving in this case that for a coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ there exists a map $\alpha : \mathscr{F} \to \mathscr{G}$ for $\mathscr{G} \in D^+_{\mathrm{coh}}(\mathscr{X})$ such that α is an isomorphism over a dense open in $\mathscr{X}$ and the corollary is known for $\mathscr{G}$.

Let $p : X \to \mathscr{X}$ be a proper surjection from a scheme X (the existence of such a p is the main theorem in [4]), so $X^\bullet$ is a simplicial proper algebraic space over A. Let $p^\bullet : X^\bullet \to \mathscr{X}$ be the structure map. Since $\mathscr{X}$ is reduced, there exists a dense open $\mathscr{U} \subseteq \mathscr{X}$ such that $X_{\mathscr{U}} = p^{-1}(\mathscr{U})$ is flat over $\mathscr{U}$, so $X^\bullet_{\mathscr{U}} \to \mathscr{U}$ is a faithfully flat proper simplicial algebraic space. Coherence of higher direct images for algebraic spaces, coupled with cohomological descent, ensures that

$$\mathscr{G} = \mathbf{R}p^\bullet_* p^{\bullet *}(\mathscr{F}) \in D^+(\mathscr{X})$$

has coherent homology sheaves and that the canonical map $\alpha : \mathscr{F} \to \mathscr{G}$ restricts to an isomorphism in $D^+_{\mathrm{coh}}(\mathscr{U})$. Thus, it suffices to prove that the corollary holds for $\mathscr{G}$.

Using cohomological descent spectral sequences, the known truth of the corollary for the proper algebraic spaces X^n over A ensures that the natural map

$$\widehat{A} \otimes \mathbf{R}\Gamma(X^\bullet, p^{\bullet *}\mathscr{F}) \to \mathbf{R}\Gamma(\widehat{X}^\bullet, (p^{\bullet *}\mathscr{F})^\wedge)$$

is an isomorphism. Likewise, the known truth of the theorem for the representable proper 1-morphisms $p^n : X^n \to \mathscr{X}$ ensures that the natural map

$$(\mathbf{R}p^\bullet_*(p^{\bullet *}\mathscr{F}))^\wedge \to \mathbf{R}\widehat{p^\bullet}_*(p^{\bullet *}\mathscr{F})^\wedge$$

in $D^+(\widehat{\mathscr{X}})$ is an isomorphism. Hence, from the commutative diagram

$$\begin{array}{ccccc}
\widehat{A}\otimes_A \mathbf{R}\Gamma(\mathscr{X}, \mathbf{R}p^\bullet_* p^{\bullet *}\mathscr{F}) & \xrightarrow{\simeq} & \widehat{A}\otimes_A \mathbf{R}\Gamma(X^\bullet, p^{\bullet *}\mathscr{F}) & & \\
\downarrow & & & \searrow^{\simeq} & \\
\mathbf{R}\Gamma(\widehat{\mathscr{X}}, (\mathbf{R}p^\bullet_*(p^{\bullet *}\mathscr{F}))^\wedge) & \xrightarrow[\simeq]{} & \mathbf{R}\Gamma(\widehat{\mathscr{X}}, \mathbf{R}\widehat{p^\bullet}_*(p^{\bullet *}\mathscr{F})^\wedge) & \xrightarrow[\simeq]{} & \mathbf{R}\Gamma(\widehat{X}^\bullet, (p^{\bullet *}\mathscr{F})^\wedge)
\end{array}$$

(in which the top row and second map on the bottom row are Leray isomorphisms) we conclude that the left side is an isomorphism. This is exactly the corollary for $\mathscr{G}$, and so concludes the proof of the theorem and the corollary.

4. The existence theorem on stacks

The main theorem is:

Theorem 4.1. *Let A be an adic noetherian ring and $f : \mathscr{X} \to \operatorname{Spec} A$ a proper 1-morphism. Let I be an ideal of definition of A. The functor*

$$\operatorname{Coh}(\mathscr{X}) \to \operatorname{Adic}_{I\cdot\mathscr{O}_{\mathscr{X}}}(\mathscr{X}) = \operatorname{Coh}(\widehat{\mathscr{X}})$$

defined by $\mathscr{F} \mapsto (\mathscr{F}/I^{n+1}\cdot\mathscr{F})$ is an equivalence of categories.

Theorem 4.1 for schemes (resp. algebraic spaces) is the proved existence theorem of Grothendieck (resp. Knutson) for proper schemes (resp. proper algebraic spaces) over an adic affine noetherian base. The general case will be deduced from the case of algebraic spaces.

Lemma 4.2. *Let $\mathscr{X}$ be a locally noetherian stack.*

(1) *For $\mathscr{F}, \mathscr{G} \in \operatorname{Coh}(\mathscr{X})$, the $\mathscr{O}_{\mathscr{X}}$-modules $\mathscr{E}xt^m_{\mathscr{X}}(\mathscr{F}, \mathscr{G})$ on $\mathscr{X}_{\text{lis-ét}}$ are coherent.*

(2) *Let $X_0 \subseteq |\mathscr{X}|$ a closed subset and $\widehat{\mathscr{X}}$ the corresponding completion. There is a unique map*

$$(\mathscr{E}xt^\bullet_{\mathscr{X}}(\mathscr{F}, \mathscr{G}))^\wedge \to \mathscr{E}xt^\bullet_{\widehat{\mathscr{X}}}(\widehat{\mathscr{F}}, \widehat{\mathscr{G}}) \tag{4.1}$$

in $\operatorname{Mod}(\mathscr{O}_{\mathscr{X}})$, for arbitrary $\mathscr{F}, \mathscr{G} \in \operatorname{Mod}(\mathscr{O}_{\mathscr{X}})$, such that it is δ-bifunctorial and coincides with the evident adjunction map in degree 0. When $\mathscr{F}$ is coherent, these maps are isomorphisms.

Proof. Argue as in [1, 0_{III}, 12.3.3, 12.3.4]. ■

Lemma 4.3. *Let $\mathscr{X}$ be a stack equipped with a proper 1-morphism $f : \mathscr{X} \to \operatorname{Spec} A$ where A is an adic noetherian ring. Let $\widehat{\mathscr{X}}$ be the completion of $\mathscr{X}$ along the zero locus on $|\mathscr{X}|$ for an ideal of definition of A.*

For coherent $\mathscr{F}$ and $\mathscr{G}$ on $\mathscr{X}$, the natural map of A-modules

$$\operatorname{Ext}^m_{\mathscr{X}}(\mathscr{F}, \mathscr{G}) \to \operatorname{Ext}^m_{\widehat{\mathscr{X}}}(\widehat{\mathscr{F}}, \widehat{\mathscr{G}}) \tag{4.2}$$

is an isomorphism for all $m \geq 0$.

Proof. Use affine formal GAGA (Corollary 3.2), Lemma 4.2, and the local-to-global Ext spectral sequence, as in [1, III_1, 4.5.2]. ■

Lemma 4.3 with $m = 0$ says that the completion functor $\mathrm{Coh}(\mathscr{X}) \to \mathrm{Coh}(\widehat{\mathscr{X}})$ is fully faithful. Objects in the essential image of this functor are called *algebraizable*. We must prove that all $\mathscr{F}$ in $\mathrm{Coh}(\widehat{\mathscr{X}})$ are algebraizable. Since (4.1) is δ-functorial, Lemma 4.3 for $m = 1$ says that algebraizability is stable under extensions (since $\mathrm{Coh}(\mathscr{X})$ is stable under extensions in $\mathrm{Mod}_{\mathscr{X}_{\text{lis-ét}}}(\mathscr{O}_{\mathscr{X}})$).

We shall argue by noetherian induction on the closed substacks of $\mathscr{X}$. The case of the substack $\emptyset$ is obvious, so we can assume that $\mathscr{X}$ is nonempty and that the existence theorem holds for any closed substack $j : \mathscr{Y} \hookrightarrow \mathscr{X}$ defined by a nonzero coherent ideal $\mathscr{J}$. In particular, pushforwards from $\mathrm{Coh}(\widehat{\mathscr{Y}})$ are algebraizable on $\mathscr{X}$ for any such $\mathscr{Y}$ because $\widehat{j}_*(\widehat{\mathscr{G}}) = (j_*\mathscr{G})^\wedge$ for $\mathscr{G} \in \mathrm{Coh}(\mathscr{Y})$. Since exactness of completion implies $(j_*(\mathscr{O}_{\mathscr{Y}}))^\wedge = \mathscr{O}_{\widehat{\mathscr{X}}}/\mathscr{J} \cdot \mathscr{O}_{\widehat{\mathscr{X}}}$, by Theorem 2.3 and the full faithfulness of $\mathrm{Coh}(\mathscr{X}) \to \mathrm{Coh}(\widehat{\mathscr{X}})$ we see that the category $\mathrm{Coh}(\widehat{\mathscr{Y}})$ is identified with the category of $\mathscr{J}$-torsion objects in $\mathrm{Coh}(\widehat{\mathscr{X}})$, and so by noetherian induction the $\mathscr{J}$-torsion objects in $\mathrm{Coh}(\widehat{\mathscr{X}})$ are algebraizable for all nonzero coherent ideals $\mathscr{J}$ on $\mathscr{X}$. As an immediate application of this fact, to prove that a given $\mathscr{F}$ in $\mathrm{Coh}(\widehat{\mathscr{X}})$ is algebraizable we may filter $\mathscr{F}$ using powers of the nilradical to reduce to the general case when $\mathscr{X}$ is reduced.

Let $p : X \to \mathscr{X}$ be a proper surjection with X a scheme, so X is proper over A. The fiber square $X^2 = X \times_{\mathscr{X}} X$ is a proper algebraic space over A; let $q : X^2 \to \mathscr{X}$ be the canonical 1-morphism. We have already noted that Theorem 4.1(2) is proved for schemes and algebraic spaces, so it is satisfied for the proper A-scheme X and the proper algebraic space X^2 over A. Hence, the pullbacks $\widehat{p}^*(\mathscr{F})$ and $\widehat{q}^*(\mathscr{F})$ in $\mathrm{Coh}(\widehat{X})$ and $\mathrm{Coh}(\widehat{X}^2)$ are algebraizable. Note that we use the ringed topoi $\widehat{X}$ and $\widehat{X}^2$, not the formal algebraic spaces $\widehat{X}_{\text{formal}}$ and $\widehat{X}^2_{\text{formal}}$. By Theorem 3.1, $\widehat{p}_*(\widehat{p}^*(\mathscr{F}))$ and $\widehat{q}_*(\widehat{q}^*(\mathscr{F}))$ in $\mathrm{Coh}(\widehat{\mathscr{X}})$ are algebraizable (since p and q are representable 1-morphisms, this step only uses Theorem 3.1 in the representable case, for which the essential content is Theorem 2.3 and hence Theorem 1.5). Thus, the kernel $\mathscr{F}' \stackrel{\text{def}}{=} \ker(\widehat{p}_*(\widehat{p}^*(\mathscr{F})) \to \widehat{q}_*(\widehat{q}^*(\mathscr{F})))$ in $\mathrm{Coh}(\widehat{\mathscr{X}})$ is algebraizable, where we are forming the kernel of the difference of the two "pullback" maps. The definition of $\mathscr{F}'$ is given in terms of pushforwards and pullbacks between maps of ringed topoi, so there is an evident morphism $\mathscr{F} \to \mathscr{F}'$ in $\mathrm{Coh}(\widehat{\mathscr{X}})$.

Enhancing the map $\mathscr{F} \to \mathscr{F}'$ to an exact sequence

$$0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{F}' \to \mathscr{G} \to 0$$

in the abelian category $\mathrm{Coh}(\widehat{\mathscr{X}})$, if we can prove that $\mathscr{K}$ and $\mathscr{G}$ are algebraizable then $\ker(\mathscr{F}' \to \mathscr{G})$ is algebraizable and hence from the resulting short exact sequence

$$0 \to \mathscr{K} \to \mathscr{F} \to \ker(\mathscr{F}' \to \mathscr{G}) \to 0$$

in $\mathrm{Coh}(\widehat{\mathscr{X}})$ it will follow (by stability of algebraizability under extensions in $\mathrm{Coh}(\widehat{\mathscr{X}})$) that $\mathscr{F}$ is algebraizable. To prove the algebraizability of $\mathscr{K}$ and $\mathscr{G}$, in view of our noetherian induction hypothesis it suffices to prove that each of $\mathscr{K}$ and $\mathscr{G}$ is each killed by a nonzero coherent ideal on $\mathscr{X}$.

Now we finally use the fact that $\mathscr{X}$ is reduced: the finite type 1-morphism $p : X \to \mathscr{X}$ must be flat over some dense open $\mathscr{V}$ in $\mathscr{X}$. Let $\mathscr{J}$ be a coherent ideal such that

$$|\mathrm{supp}(\mathscr{O}_{\mathscr{X}}/\mathscr{J})| = |\mathscr{X}| - |\mathscr{V}|,$$

so $\mathscr{J}^r \neq 0$ for all $r \geq 1$ (since $\mathscr{X}$ is nonempty). It therefore suffices to show that each of $\mathscr{K}$ and $\mathscr{G}$ in $\mathrm{Coh}(\widehat{\mathscr{X}})$ is killed by a power of $\mathscr{J}$.

Let $U = \operatorname{Spec} B \to \mathscr{X}$ be a smooth covering by an affine scheme, and let U_0 be the preimage of $\mathscr{X}_0$ in U (that is, U_0 is the fiber of U over $\operatorname{Spec} A/I$). Since $\mathscr{X}$ is proper over $\operatorname{Spec} A$ and $U \to \mathscr{X}$ is open, any open neighborhood W of U_0 in U has image in $\mathscr{X}$ that is open and contains $\mathscr{X}_0$, whence *by the adic property of* A it follows that W must surject onto $|\mathscr{X}|$ (as the complement of the image of W in $|\mathscr{X}|$ is a closed set whose closed image in $\operatorname{Spec} A$ is disjoint from $\operatorname{Spec} A/I$ and thus is empty). The pullback of $\mathscr{F}$ to the (lisse-étale) ringed topos $\widehat{U}$ is an object in $\operatorname{Coh}(\widehat{U}) = \operatorname{Coh}(\operatorname{Spf} \widehat{B})$ (Remark 1.6), so let M be the finite $\widehat{B}$-module arising from $\mathscr{F}$; here, $\widehat{B}$ is the IB-adic completion of B.

Since formation of the flat locus of a 1-morphism of stacks is compatible with faithfully flat base change, it remains (by Theorem 1.5 and formal GAGA) to apply the following lemma to the algebraic space pullback $p' : X_U \to U$ of the surjective $p : X \to \mathscr{X}$ by the *fppf* base change $U = \operatorname{Spec} B \to \mathscr{X}$:

Lemma 4.4. *Let $p : X \to \operatorname{Spec} B$ be a proper algebraic space surjecting onto an affine noetherian scheme and let $q : X^2 \to \operatorname{Spec} B$ be the fiber square of p. Let I be an ideal in B and let M a finite module over the I-adic completion $\widehat{B}$. Let $V \subseteq \operatorname{Spec} B$ be a Zariski-open over which p is flat, and let J be an ideal of B such that $|\operatorname{Spec} B/J| = |\operatorname{Spec} B| - V$.*

Let $\mathfrak{X} = \widehat{X}_{\mathrm{formal}}$ with its étale topology, and let

$$\widehat{p} : \mathfrak{X} \to \operatorname{Spf} \widehat{B}, \ \ \widehat{q} : \mathfrak{X} \times_{\operatorname{Spf} \widehat{B}} \mathfrak{X} \to \operatorname{Spf} \widehat{B}$$

be the canonical surjective structure maps of formal separated noetherian algebraic spaces (with the étale topology). Consider the map of finite $\widehat{B}$-modules

$$M \to \ker(\mathrm{H}^0(\mathfrak{X}, \widehat{p}^*(M^{\Delta}_{\text{ét}})) \to \mathrm{H}^0(\mathfrak{X} \times_{\operatorname{Spf} \widehat{B}} \mathfrak{X}, \widehat{q}^*(M^{\Delta}_{\text{ét}}))) \tag{4.3}$$

as a map of quasi-coherent sheaves on $\operatorname{Spec} B$. There exists a Zariski-open neighborhood W of $\operatorname{Spec} B/I$ in $\operatorname{Spec} B$ over which the kernel and cokernel of (4.3) are killed by powers of J.

Proof. By quasi-compactness of $\operatorname{Spec} B/I$ and the finiteness of the modules in (4.3) over the noetherian ring $\widehat{B}$, standard smearing-out arguments allow us to localize the problem at points of $\operatorname{Spec} B/I$. That is, it suffices to show that for each $z \in \operatorname{Spec} B/I$ there is a power of J that kills the kernel and cokernel of the algebraic localization of (4.3) at $z \in \operatorname{Spec} B$. The maximal-adic completion $B_z^\wedge$ is faithfully flat over B_z and is $IB_z^\wedge$-adically separated and complete, so by making the base changes by $\operatorname{Spec} B_z^\wedge \to \operatorname{Spec} B$ and $\operatorname{Spf} B_z^\wedge \to \operatorname{Spf} \widehat{B}$ where $B_z^\wedge$ is given the $IB_z^\wedge$-adic topology we are reduced to the case when B is I-adically separated and complete (so $B = \widehat{B}$). Of course, this reduction step uses the obvious fact that formation of global sections of a coherent sheaf on a formal separated noetherian algebraic space over $\operatorname{Spf} \widehat{B}$ commutes with the formally flat affine base change $\operatorname{Spf} B_z^\wedge \to \operatorname{Spf} \widehat{B}$

In the case $B = \widehat{B}$, formal GAGA for algebraic spaces and formal completions thereof identifies (4.3) with the natural map of finite B-modules

$$M \to \ker(\mathrm{H}^0(X, p^*(\widetilde{M}_{\text{ét}})) \to \mathrm{H}^0(X \times_{\operatorname{Spec} B} X, q^*(\widetilde{M}_{\text{ét}}))).$$

In other words, this is the map of coherent sheaves

$$\mathscr{F} \to \ker(p_* p^*(\mathscr{F}) \to q_*(q^*(\mathscr{F}))) \tag{4.4}$$

on $(\operatorname{Spec} B)_{\text{ét}}$, using $\mathscr{F} = \widetilde{M}_{\text{ét}}$. Since $p : X \to \operatorname{Spec} B$ is surjective and is flat over the Zariski-open $V \subseteq \operatorname{Spec} B$, the restriction $p^{-1}(V) \to V$ is an *fppf* morphism. Thus, *fppf* descent theory for quasi-coherent sheaves on algebraic spaces ensures that (4.4) restricts to an isomorphism over V and hence the kernel and cokernel of (4.4) restrict to 0 on V. Since the kernel and cokernel of (4.4) are coherent, each must therefore be annihilated by a power of the ideal of definition J of the Zariski-closed complement of V in $\operatorname{Spec} B$. ■

Using Stein factorization for proper morphisms [3, 14.2.8], we conclude via the theorem on formal functions (exactly as for schemes) that the Zariski connectedness theorem holds:

Corollary 4.5. *If $f : \mathscr{X} \to \mathscr{Y}$ is a proper morphism between locally noetherian Artin stacks then its Stein factorization has geometrically connected fibers.*

For both schemes and algebraic spaces, there is a stronger form of the existence theorem in which the properness hypothesis on the morphism $f : \mathscr{X} \to \operatorname{Spec} A$ is relaxed to merely being separated and locally of finite type but the coherent sheaves on $\mathscr{X}$ and the $\mathscr{X}_n$'s are required to have proper support. In more intrinsic language, for any $\mathscr{F}$ in $\operatorname{Coh}(\widehat{\mathscr{X}})$ and any open ideal I in A the associated coherent sheaf $\mathscr{F} \bmod I$ has support given by a closed subset in $|\mathscr{X}_0|$ that is independent of I. We call this closed substack (say, with reduced structure) the *support* of $\mathscr{F}$. The stronger version of the existence theorem also carries over to Artin stacks:

Corollary 4.6. *Let $f : \mathscr{X} \to \operatorname{Spec} A$ be separated and locally finite type, with $\mathscr{X}$ an Artin stack and A an adic noetherian ring. The completion functor $\operatorname{Coh}(\mathscr{X}) \to \operatorname{Coh}(\widehat{\mathscr{X}})$ restricts to an equivalence between the full subcategories consisting of objects with A-proper support.*

Proof. Since $\mathscr{X}$ is covered by quasi-compact opens, it is clearly sufficient to prove the corollary in the quasi-compact case. That is, we can assume $\mathscr{X}$ is separated and of finite type over A.

We first verify full faithfulness. Consider coherent $\mathscr{F}$ and $\mathscr{G}$ on $\mathscr{X}$ with proper support, and let $\widehat{\mathscr{F}}$ and $\widehat{\mathscr{G}}$ be the associated completions. Let $\mathscr{I}$ and $\mathscr{J}$ be coherent ideals on $\mathscr{X}$ that cut out the supports of $\mathscr{F}$ and $\mathscr{G}$. By quasi-compactness of the supports, we can replace $\mathscr{I}$ and $\mathscr{J}$ with suitable powers to arrange that $\mathscr{I}$ kills $\mathscr{F}$ and $\mathscr{J}$ kills $\mathscr{G}$. Thus, $\mathscr{I} \cdot \mathscr{J}$ kills $\mathscr{F}$ and $\mathscr{G}$, so each of $\mathscr{F}$ and $\mathscr{G}$ is a pushforward of a coherent sheaf on the closed substack $\mathscr{Z} \subseteq \mathscr{X}$ cut out by $\mathscr{I} \cdot \mathscr{J}$; this substack is A-proper since $|\mathscr{Z}| \subseteq |\mathscr{X}|$ is the union of the supports of the A-proper closed substacks cut out by $\mathscr{I}$ and $\mathscr{J}$. The completions $\widehat{\mathscr{F}}$ and $\widehat{\mathscr{G}}$ lie in the full subcategory $\operatorname{Coh}(\widehat{\mathscr{Z}}) \subseteq \operatorname{Coh}(\widehat{\mathscr{X}})$, and so the full faithfulness follows from the settled proper case applied to $\mathscr{Z}$.

For essential surjectivity, by the settled proper case it suffices to show that if $\mathscr{F} \in \operatorname{Coh}(\widehat{\mathscr{X}})$ has proper support then it lies in $\operatorname{Coh}(\widehat{\mathscr{Z}})$ for some A-proper closed substack $\mathscr{Z} \subseteq \mathscr{X}$. Let $p : X \to \mathscr{X}$ be a proper surjection from a scheme, and consider $\widehat{p}^*(\mathscr{F}) \in \operatorname{Coh}(\widehat{X})$. This has support given by the p-preimage of the proper support of $\mathscr{F}$, so it is A-proper since p is proper. Hence, by the known scheme case there is a coherent sheaf on X with A-proper support that algebraizes $\widehat{p}^*(\mathscr{F})$. Let $Z \subseteq X$ be the A-proper underlying reduced scheme for the support this algebraization, and let $\mathscr{Z} \subseteq \mathscr{X}$ be the closed image of Z under the proper map p, say given its reduced structure (so there is a surjective morphism $Z \to \mathscr{Z}$ since Z is reduced, and thus $Z \subseteq p^{-1}(\mathscr{Z})$; it follows that a power of the coherent ideal of $p^{-1}(\mathscr{Z})$ in $\mathscr{X}$ kills the algebraization of $\widehat{p}^*(\mathscr{F})$). By the same argument as for schemes, since the A-proper Z surjects onto the A-separated $\mathscr{Z}$ it follows that $\mathscr{Z}$ is A-proper. Let $\mathscr{I}$ be the radical coherent ideal sheaf on $\mathscr{X}$ that cuts out $\mathscr{Z}$. We shall prove that some power of $\mathscr{I}$ kills $\mathscr{F} \in \operatorname{Coh}(\widehat{\mathscr{X}})$, and so $\mathscr{F}$ is a pushforward from $\operatorname{Coh}(\widehat{\mathscr{Z}_n})$ for some large n, with $\mathscr{Z}_n \subseteq \mathscr{X}$ the A-proper closed substack cut out by $\mathscr{I}^n$. This will complete the proof, as we have explained above.

It is convenient to now generalize our problem so that it is easier to localize. Let $p : Y \to \mathscr{Y}$ be a proper surjection from an algebraic space onto a noetherian Artin stack, and consider $\mathscr{F} \in \mathrm{Coh}(\widehat{\mathscr{Y}})$, where $\widehat{\mathscr{Y}}$ is the completion along a closed subset Σ in $|\mathscr{Y}|$. Let $\mathscr{Z} \subseteq \mathscr{Y}$ be a closed substack whose preimage $Z \subseteq Y$ has defining (radical) ideal $\mathscr{J}$ such that some power of $\mathscr{J}$ kills $\widehat{p}^*(\mathscr{F}) \in \mathrm{Coh}(\widehat{Y})$ (with $\widehat{Y}$ denoting the completion of Y along $p^{-1}(\Sigma) \subseteq |Y|$). We claim that some power of the radical ideal that defines $\mathscr{Z}$ in $\mathscr{Y}$ kills $\mathscr{F}$. The hypotheses are preserved by any base change along a 1-morphism $\mathscr{Y}' \to \mathscr{Y}$ with $\mathscr{Y}'$ another noetherian Artin stack. Hence, by using a smooth cover of $\mathscr{Y}$ by an affine scheme we reduce to the case when $\mathscr{Y} = \mathrm{Spec}\, B$ is an affine scheme. By Chow's lemma for algebraic spaces we can also assume that the algebraic space Y is a scheme. Let $K \subseteq B$ be the radical ideal corresponding to $\Sigma \subseteq |\mathscr{Y}|$ and let $\widehat{B}$ be the completion of B along K, so $\mathscr{F}$ corresponds to a finite $\widehat{B}$-module M. Let I be the ideal of $\mathscr{Z}$ in $\mathscr{Y} = \mathrm{Spec}\, B$, so by hypothesis some I^n kills $\widehat{p}^*(\mathscr{F}) \in \mathrm{Coh}(\widehat{Y})$. Thus, the pullback of $\widetilde{M}$ along the proper surjective morphism $\pi : Y \otimes_B \widehat{B} \to \mathrm{Spec}\, \widehat{B}$ is killed by I^n on all infinitesimal neighborhoods of the fiber over $\mathrm{Spec}\, B/K$, so by properness of π the ideal I^n kills the pullback of $\widetilde{M}$ to $Y \otimes_B \widehat{B}$. Hence, by surjectivity of π, the support of M in $\mathrm{Spec}\, \widehat{B}$ is contained in the zero locus of I^n. Some power of I^n therefore kills M, and this resulting power of I also kills $\mathscr{F} \in \mathrm{Coh}(\widehat{\mathscr{Y}}) = \mathrm{Coh}(\mathrm{Spf}(\widehat{B}))$. ■

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